Comments about the paper entitled “A generalized boundary integral equation for isotropic heat conduction with spatially varying thermal conductivity” by A.J. Kassab and E. Divo.

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Comments about the paper entitled "A generalized boundary integral equation for isotropic heat conduction with spatially varying thermal conductivity" by A.J. Kassab and E. Divo*

Marc Bonnet†, Massimo Guiggiani‡

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Abstract

An integral formulation for heat conduction problems in non-homogeneous media has recently been proposed in [1]. The goal of this communication is to revisit and clarify two key features of the formulation of [1]. First, the contention that the integral equation formulation proposed in [1] does not possess the desired boundary-only character is made and substantiated; it is shown in particular that Eq. (10) therein does not hold due to the fact that a crucial requirement for the fundamental solution, Eq. (5c-d) of [1] is actually not met. Second, the limiting process associated with a vanishing neighbourhood in connexion with the particular kernel function used in [1] is revisited.

The purpose of the paper under discussion [1] was to establish a boundary-only integral formulation for heat conduction with isotropic but spatially varying conductivity $k(\boldsymbol{x})$ ($\boldsymbol{x} = (x, y)$: field point), i.e. for temperature distributions T such that

$$\operatorname{div}\left[k(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\right] = 0 \tag{1}$$

In order to do so, a variant kind of fundamental solution, denoted $E(\boldsymbol{x}, \boldsymbol{\xi})$, associated to a forcing term $D(\boldsymbol{x}, \boldsymbol{\xi})$ containing a singular source at a given point $\boldsymbol{\xi} = (x_i, y_i)$, is introduced:

$$\operatorname{div}\left[k(\boldsymbol{x})\boldsymbol{\nabla}E(\boldsymbol{x}, \boldsymbol{\xi})\right] = -D(\boldsymbol{x}, \boldsymbol{\xi}) \tag{2}$$

1 Examination of the sampling property

The following, crucial, requirement on the forcing term $D(\boldsymbol{x}, \boldsymbol{\xi})$ (Eqs. (5c-d) of [1]) must be met in order to get rid of the domain integral that arises in the reciprocal theorem (Ω denotes an arbitrary domain required only to enclose the source point $\boldsymbol{\xi}$):

$$\int_\Omega T(x)D(\boldsymbol{x}, \boldsymbol{\xi})\,\mathrm{d}\Omega(\boldsymbol{x}) = T(\boldsymbol{\xi})A(\boldsymbol{\xi}) \qquad \text{with} \quad A(\boldsymbol{\xi}) = \int_\Omega D(\boldsymbol{x}, \boldsymbol{\xi})\,\mathrm{d}\Omega(\boldsymbol{x}) \tag{3}$$

The above relation is the analogue, for the particular fundamental solution at hand, of the *sampling property* of the Dirac distribution (the term *sifting property* was used in [1]). However, the kernel $E(\boldsymbol{x}, \boldsymbol{\xi})$ obtained by the authors, i.e. (Eq. (23) of [1]):

$$E(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{2\pi}\int \frac{\mathrm{d}r}{r\tilde{k}(r; x_i, y_i)} \tag{4}$$

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† Laboratoire de Mécanique des Solides (UMR CNRS 7649), Ecole Polytechnique, 91128 Palaiseau cedex, France

‡ Dipartimento di Matematica, Università degli Studi di Siena, Via del Capitano 15, 53100 Siena, Italy

where (in 2D) $(x,y) = (x_i + r\cos\theta, y_i + r\sin\theta)$ and having introduced for convenience the 'averaged' conductivity $\tilde{k}$:

$$\tilde{k}(r; x_i, y_i) = \frac{1}{2\pi}\int_0^{2\pi} k(r,\theta; x_i, y_i)\mathrm{d}\theta \tag{5}$$

happens to violate the crucial requirement (3). Indeed, the kernel given by Eq. (4) is readily seen to solve the equation

$$\mathrm{div}\,[\tilde{k}\boldsymbol{\nabla}E] = -\delta(\boldsymbol{x}-\boldsymbol{\xi}) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\tilde{k}\frac{\partial E}{\partial r}\right] \tag{6}$$

(the second equality holds because $\partial E/\partial\theta = 0$). Hence, by subtracting the above equation from Eq. (2), one gets the following expression for the forcing term D:

$$D(\boldsymbol{x},\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}) + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\tilde{k}-k)\frac{\partial E}{\partial r}\right] = \delta(\boldsymbol{x}-\boldsymbol{\xi}) + \frac{1}{2\pi r}\frac{\partial}{\partial r}\left[1-\frac{k}{\tilde{k}}\right] \equiv \delta(\boldsymbol{x}-\boldsymbol{\xi}) + \Delta D \tag{7}$$

Except when the material is homogeneous, the complementary term ΔD is therefore in general nonzero at $x \neq \xi$, ie outside the source point ξ; this is indeed obvious in some of the examples of the paper under discussion (see Eqs. (40,46) and Figs. 3,5,12). In fact, for sufficiently smooth conductivities, $k - \tilde{k} = O(r)$ so that ΔD is even *nonsingular* at $\boldsymbol{x} = \boldsymbol{\xi}$. But then, it is impossible for properties (3) to hold. In fact, using the definition of $A(\boldsymbol{\xi})$ for a given domain Ω, one has:

$$\int_\Omega T(\boldsymbol{x})D(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\Omega = A(\boldsymbol{\xi})T(\boldsymbol{\xi}) + \int_\Omega [T(\boldsymbol{x}) - T(\boldsymbol{\xi})]\Delta D(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\Omega \equiv A(\boldsymbol{\xi})T(\boldsymbol{\xi}) + J(\boldsymbol{\xi}) \tag{8}$$

and property (3) would mean that the integral $J(\boldsymbol{\xi})$ in the above formula, which measures the absolute error in the realization of (3), vanishes for any domain shape and temperature distribution, which is false by Haar's lemma since D is not identically zero outside the source point. For example, Eq. (5c-d) of [1] for the same function T (which incidentally is required to have compact support) but two different domains $\Omega_1 \subset \Omega_2$ would lead to:

$$\int_{\Omega_2\setminus\Omega_1} T(\boldsymbol{x})D(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\Omega(x) = T(\boldsymbol{\xi})[A_2(\boldsymbol{\xi}) - A_1(\boldsymbol{\xi})]$$

and a contradiction arises. From the above equation, the integral in the l.h.s. should, apart from the geometrical arrangement of ξ, Ω_1, Ω_2, depend only on the value of T *at the source point* $\boldsymbol{\xi}$. But since $\Omega_2 \setminus \Omega_1$ does *not* contain ξ, the above equation should hold true for any extension of T outside $\Omega_2 \setminus \Omega_1$, which is absurd. In other words, as soon as $D(\boldsymbol{x},\boldsymbol{\xi})$ is nonzero over $\Omega_2 \setminus \Omega_1$, the integral

$$\int_{\Omega_2\setminus\Omega_1} T(\boldsymbol{x})D(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\Omega(x)$$

cannot depend on T only through the value $T(\boldsymbol{\xi})$. Thus, Eq. (3) is in general not true. As a consequence, the two members of the integral equation (10) in [1] are in general not equal.

2 Limiting process for vanishing neighbourhood

Let $\boldsymbol{\xi}$ be a fixed point on the boundary $\partial\Omega$ of a two-dimensional domain Ω. We consider an exclusion neighbourhood $v_\varepsilon(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$, of radius $\leq \varepsilon$ (Figure 1). For any $\varepsilon > 0$, $\boldsymbol{\xi}$ is always an external point for the domain $\Omega_\varepsilon(\boldsymbol{\xi}) = \Omega \setminus v_\varepsilon(\boldsymbol{\xi})$ whose boundary $\partial\Omega_\varepsilon$ is given by

$$\partial\Omega_\varepsilon = (\partial\Omega - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon + s_\varepsilon,$$

where $s_\varepsilon = \Omega \cap \partial v_\varepsilon$, $e_\varepsilon = \partial\Omega \cap \bar{v}_\varepsilon$, and $\Gamma_\varepsilon = \partial\Omega - e_\varepsilon$ ($\bar{v}_\varepsilon$ is the closure of v_ε).

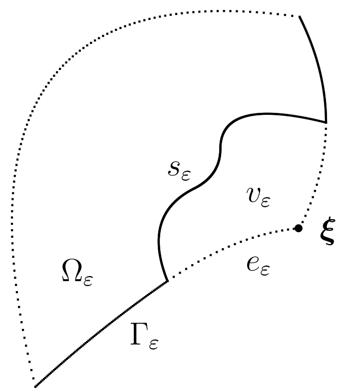

Figure 1: Exclusion of the singular point $\boldsymbol{\xi}$ by a vanishing neighbourhood v_ε.

The singular integral equation for heat conduction problems (T: unknown temperature field) in non-homogeneous media can then be sought as the limiting case, for $\varepsilon \to 0$, of the integral identity

$$\int_{\Gamma_\varepsilon + s_\varepsilon} E(\boldsymbol{x},\boldsymbol{\xi}) k(\boldsymbol{x}) \frac{\partial T}{\partial n}(\boldsymbol{x})\, dS_x - \int_{\Gamma_\varepsilon + s_\varepsilon} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi}) T(\boldsymbol{x})\, dS_x - \int_{\Omega_\varepsilon} T(\boldsymbol{x}) D(\boldsymbol{x},\boldsymbol{\xi})\, \mathrm{d}\Omega(\boldsymbol{x}) = 0 \quad (9)$$

where $E(\boldsymbol{x},\boldsymbol{\xi})$ is the fundamental solution defined by Eqs. (4,5).

Source point interior to the domain. First, consider an internal source point $\boldsymbol{\xi} \notin \partial\Omega$. Then s_ε is a closed surface, and one has:

$$\int_{\Omega_\varepsilon} D(\boldsymbol{x},\boldsymbol{\xi})\, \mathrm{d}\Omega(\boldsymbol{x}) = -\int_{s_\varepsilon} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\, dS_x - \int_{\partial\Omega} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\, dS_x \quad (10)$$

Since $\Gamma_\varepsilon = \partial\Omega$, identity (9) becomes (note that $D = \Delta D$ on Ω_ε):

$$\begin{aligned} &\int_{\partial\Omega} E(\boldsymbol{x},\boldsymbol{\xi}) k(\boldsymbol{x}) \frac{\partial T}{\partial n}(\boldsymbol{x})\, dS_x - \int_{\partial\Omega} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi}) T(\boldsymbol{x})\, dS_x \\ &+ \int_{s_\varepsilon} E(\boldsymbol{x},\boldsymbol{\xi}) k(\boldsymbol{x}) \frac{\partial T}{\partial n}(\boldsymbol{x})\, dS_x - \int_{s_\varepsilon} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi}) [T(\boldsymbol{x}) - T(\boldsymbol{\xi})]\, dS_x \\ &- \int_{\Omega_\varepsilon} [T(\boldsymbol{x}) - T(\boldsymbol{\xi})] \Delta D(\boldsymbol{x},\boldsymbol{\xi})\, \mathrm{d}\Omega(\boldsymbol{x}) + T(\boldsymbol{\xi}) \int_{\partial\Omega} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\, dS_x = 0 \end{aligned} \quad (11)$$

Assuming that T is $C^{0,\alpha}$-regular at $\boldsymbol{x} = \boldsymbol{\xi}$, the integrals over s_ε vanish in the limit $\varepsilon \to 0$ in the same way, due to the fact that the kernel E exhibits the usual $\ln r$ singularity at $\boldsymbol{x} = \boldsymbol{\xi}$. Hence, the following integral identity is obtained from Eq. (11) in the limit $\varepsilon \to 0$:

$$T(\boldsymbol{\xi}) \tilde{A}(\boldsymbol{\xi}) = \int_{\partial\Omega} E(\boldsymbol{x},\boldsymbol{\xi}) k(\boldsymbol{x}) \frac{\partial T}{\partial n}(\boldsymbol{x})\, dS_x - \int_{\partial\Omega} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi}) T(\boldsymbol{x})\, dS_x - J(\boldsymbol{\xi}) \quad (12)$$

where the free-term $\tilde{A}(\boldsymbol{\xi})$ is given by

$$\tilde{A}(\boldsymbol{\xi}) = -\int_{\partial\Omega} k(\boldsymbol{x}) \frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\, dS_x \quad (13)$$

It is worth noting (still using the $C^{0,\alpha}$ regularity assumption for T) that

$$\int_{\Omega_\varepsilon} [T(\boldsymbol{x}) - T(\boldsymbol{\xi})] D(\boldsymbol{x},\boldsymbol{\xi})\, \mathrm{d}\Omega(\boldsymbol{x}) = \int_{\Omega_\varepsilon} [T(\boldsymbol{x}) - T(\boldsymbol{\xi})] \Delta D(\boldsymbol{x},\boldsymbol{\xi})\, \mathrm{d}\Omega(\boldsymbol{x}) = J(\boldsymbol{\xi})$$

Hence, from the analysis of Sec. 1, a boundary-only representation formula for T cannot be obtained in general.

Source point on the boundary. A similar analysis can be conducted. Using the fact that

$$\int_{\Omega_\varepsilon} D(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\Omega(\boldsymbol{x}) = -\int_{\Gamma_\varepsilon+s_\varepsilon} k(\boldsymbol{x})\frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\,dS_x$$

one obtains in the limit $\varepsilon \to 0$ the following statement:

$$T(\boldsymbol{\xi})\hat{A}(\boldsymbol{\xi}) = \int_{\partial\Omega} E(\boldsymbol{x},\boldsymbol{\xi})k(\boldsymbol{x})\frac{\partial T}{\partial n}(\boldsymbol{x})\,dS_x - \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} k(\boldsymbol{x})\frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})T(\boldsymbol{x})\,dS_x - J(\boldsymbol{\xi}) \tag{14}$$

where the free-term $\hat{A}(\boldsymbol{\xi})$ is given by

$$\hat{A}(\boldsymbol{\xi}) = -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} k(\boldsymbol{x})\frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\,dS_x = -\int_{\partial\Omega} k(\boldsymbol{x})\frac{\partial E}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})\,dS_x \tag{15}$$

Here again, the domain integral is not expected to vanish. The above integral is convergent in the ordinary sense in the limit $\varepsilon \to 0$ because $\partial E/\partial n$ is proportional to $\partial r/\partial n$.

3 Discussion

From the analysis conducted in Sec. 1, the integral representation (13) and the integral equation (15) involve in general a domain term. It is thus not correct to state, as was done in [1], that recourse to the kernel E leads to boundary-only integral formulations for heat conduction problems in non-homogeneous media. However, since E is in some sense an 'averaged' version, associated with the 'averaged' conductivity $\tilde{k}$, of the (unknown) fundamental solution associated with the 'true' conductivity k, it is reasonable to expect the domain integrals in Eqs. (13) and (15) to be lower than those associated with the usual fundamental solution $G(\boldsymbol{x},\boldsymbol{\xi}) = -(2\pi k)^{-1}\ln r$ for homogeneous conductivity k.

The kernel E given by Eqs. (4,5) is not strictly speaking a fundamental solution: the conductivity $\tilde{k}$, and hence the partial differential operator in Eq. (6), depends on the source point $\boldsymbol{\xi}$. However, for a fixed choice of $\boldsymbol{\xi}$ the kernel E behave like the more usual fundamental solutions.

It is also important at this juncture to remark that using the relative accuracy of satisfaction of Eq. 8 as a measure of the exactness of is misleading. The ratio of $A(\xi)T(\xi)$ to the integral term $J(\xi)$ can assume any arbitrarily small value by adding a constant to the sought-for temperature. The reason for this behaviour is that J is insensitive to the choice of this constant. On the other hand, fact, what is under scrutiny here is the exactness of a mathematical formulation, and hence the *absolute* validity of Eq. (3).

Besides, when $k(\boldsymbol{x})$ is a polynomial in x, y (or x, y, z in 3D) involving only odd powers of the coordinates, $\tilde{k} = k(\boldsymbol{\xi})$: in this case, $E(\boldsymbol{x},\boldsymbol{\xi})$ reduces to the classical fundamental solution for a medium with a *constant* conductivity equal to $k(\boldsymbol{\xi})$. This is the case for the bi- and tri-linear conductivity distributions (Eqs. (33,36) in [1]).

The free terms $\tilde{A}(\boldsymbol{\xi})$ and $\hat{A}(\boldsymbol{\xi})$ correspond to Eqs. (8) and (30) of [1]. They depend on the source point location *and* on the domain geometry, and therefore must be numerically computed for each equation of the BEM discrete system and for each interior-point temperature evaluation requested by the user. For homogeneous media, $\tilde{A}(\boldsymbol{\xi}) = 1$ and $\hat{A}(\boldsymbol{\xi}) = 1/2$ at smooth boundary points (i.e the usual geometry-independent values are recovered as special cases) while of course all domain integrals vanish.

4 Illustrative numerical examples

In order to illustrate tha above considerations and have some quantitative idea of the approximate character of Eq. (3), two 2D examples are now considered.

The first example is analytical. We take $k = (1+x)(1+y)$ and $\Omega = \{0 \le x, y \le 2\}$. A solution to Eq. (1) is $T = (1+x)^2 - (1+y)^2$. One then finds

$$\tilde{k} = (1+x_i)(1+y_i) \qquad \text{(i.e. constant)}$$

$$\frac{\partial}{\partial r}(\tilde{k} - k) = (1+y_i)\cos\theta + (1+x_i)\sin\theta + 2r\cos\theta\sin\theta$$

To keep analytical calculations simple, we consider only source points on the horizontal median of Ω, i.e. such that $y_i = 1$. After some calculations, one finds for such source points:

$$A(x_i, 1) = 1 + \frac{1}{2\pi(1+x_i)}\left[(2-x_i)\theta_1 - x_i\theta_2 - \mathrm{Log}\frac{1+x_i^2}{5-4x_i+x_i^2}\right] \tag{16}$$

$$\begin{aligned} J(x_i, 1) = \frac{1}{\pi(1+x_i)}\Big[&3(x_i-1) + 3(2-x_i)^2\theta_1 + (13 - 11x_i + 3x_i^2)\theta_2 \\ &-(1+x_i)\pi + \frac{1}{2}\mathrm{Log}\frac{1+x_i^2}{5-4x_i+x_i^2}\Big] \end{aligned} \tag{17}$$

The functions $A(x_i, 1)$ and $J(x_i, 1)$ are plotted for $0 \le x \le 2$ in Fig. 2, where $J(x_i, 1)$ is clearly nonzero. The source distribution $D(\boldsymbol{x}, \boldsymbol{\xi})$, shown in Fig. 3 for $x_i = 0.505, y_i = 1.255$, visibly assumes nonzero values outside the source point.

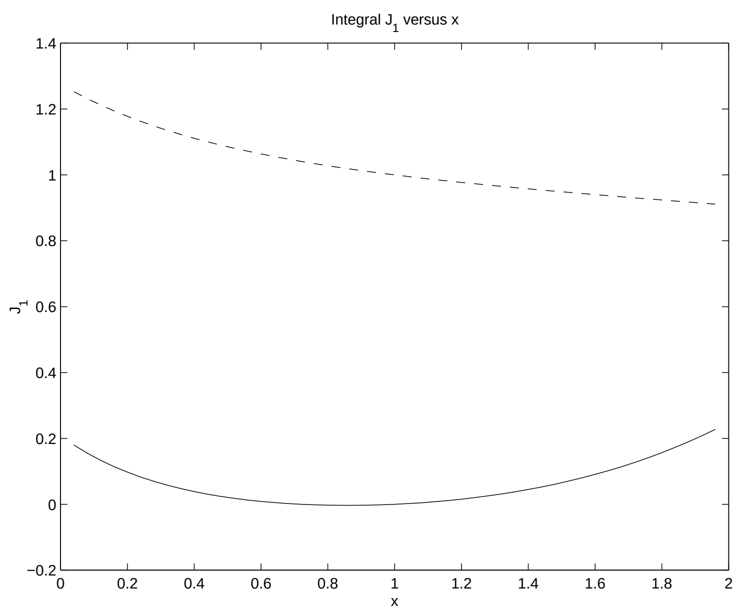

Figure 2: Example 1: variation of $A(x_i, 1)$ (dashed line) and $J(x_i, 1)$ (solid line).

The second example is numerical (it is Ex. 1 of [1]). We take $k = (1 + x/a)^3$ and $\Omega = \{0 \le x, y \le a\}$, for some length a. A solution to Eq. (1) is $T = 4a^2/(3(a+x)^2)$ (having dropped the factor $T_0 = 100$ and the additive constant). Here, one has $\tilde{k} = k(\boldsymbol{\xi}) + 3r^2(x_i + a)/(2a^3)$. The integral $J(\boldsymbol{\xi})$ has been evaluated numerically (cutting the square domain into four triangles meeting at $\boldsymbol{\xi}$, mapping each triangle onto a square and using a 200×200 point Gaussian quadrature rule on each subdomain). The values of $J(\boldsymbol{\xi})$ for $0 < x_i, y_i < a$, shown in Fig. 4, are clearly nonzero. For completeness, the values $\tilde{J}(\boldsymbol{\xi})$ taken by $J(\boldsymbol{\xi})$ using the homogeneous medium fundamental solution associated with the constant conductivity $k(\boldsymbol{\xi})$ is depicted in Fig. 5. A numerical calculation gives:

$$\left\{\int_\Omega J^2(\boldsymbol{\xi})\,\mathrm{d}\Omega(\boldsymbol{\xi})\right\}^{1/2} \approx 0.367 \times \left\{\int_\Omega \tilde{J}^2(\boldsymbol{\xi})\,\mathrm{d}\Omega(\boldsymbol{\xi})\right\}^{1/2}$$

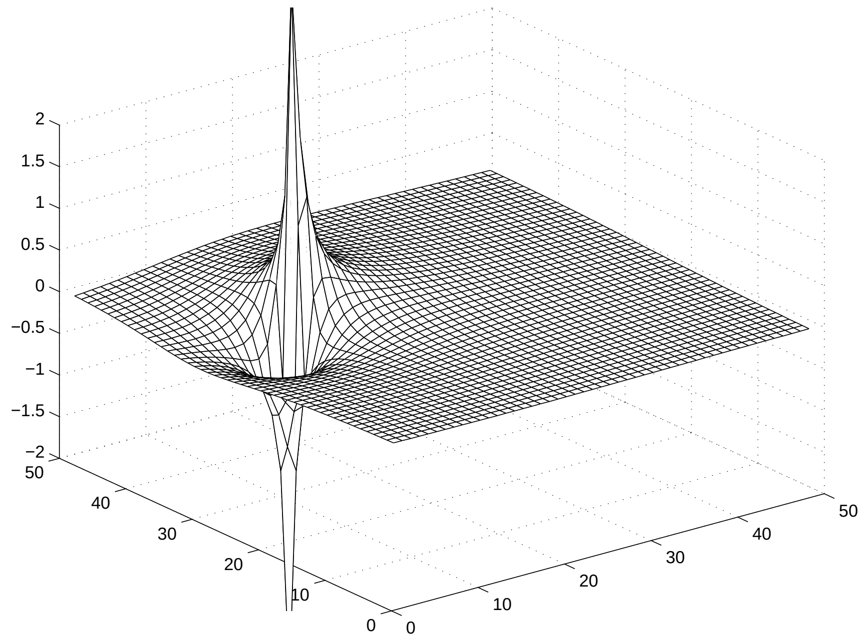

Figure 3: Example 1: forcing function $D(\boldsymbol{x}, \boldsymbol{\xi})$ for $x_i = 0.505, y_i = 1.255$ (horizontal units refer to a regular 50×50 grid used for the plot).

In other words, for this particular example, $J(\boldsymbol{\xi})$ as a function of $\boldsymbol{\xi}$ is, in a L^2-norm average sense, 0.367 times smaller than the domain integral $\tilde{J}^2(\boldsymbol{\xi})$ encountered when using the 'best' homogeneous-medium fundamental solution.

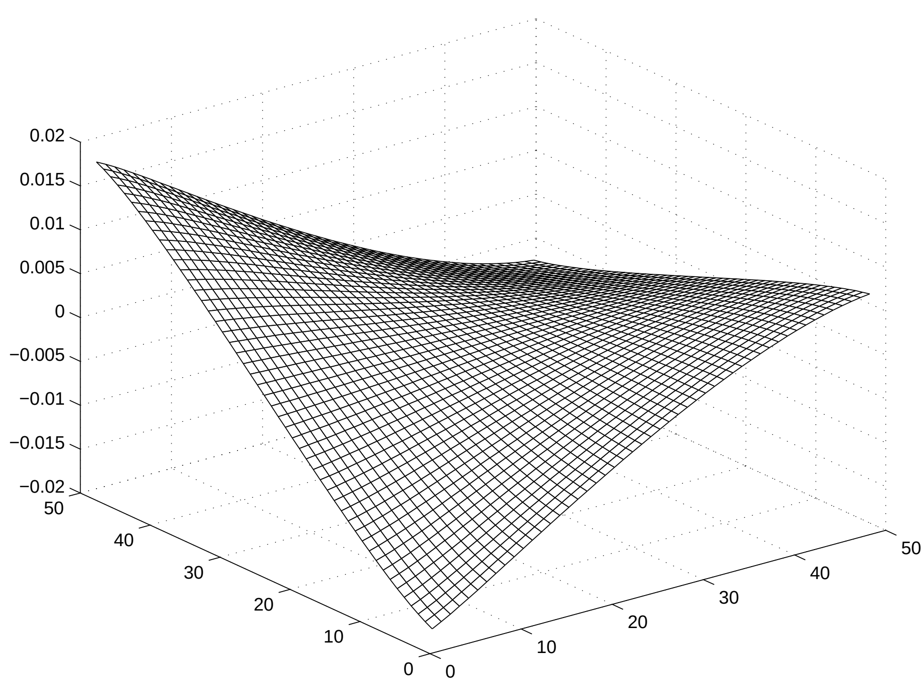

Figure 4: Example 2: variation of $J(x_i, 1)$ (horizontal units refer to a regular 50×50 grid used for the plot).

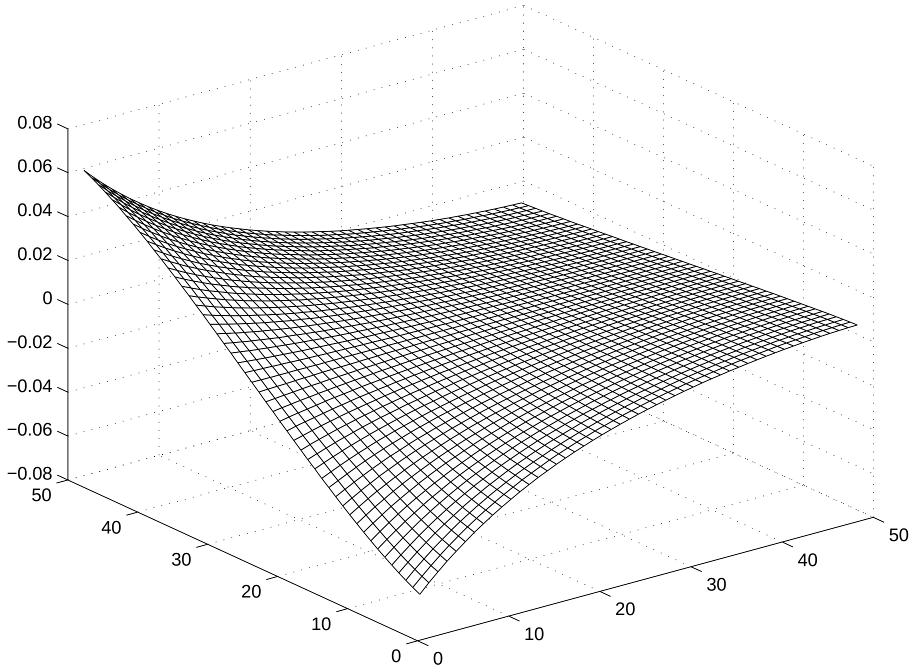

Figure 5: Example 2: variation of $\tilde{J}(x_i, 1)$ (horizontal units refer to a regular 50×50 grid used for the plot).

5 Conclusion

From the present analysis, our contention is that the approach proposed in [1], contrarily to the claim made in that paper, does not lead to an integral equation without domain integrals. In particular, the integral equation, Eq. (10) of [1], is found to be only approximately, not exactly, satisfied. The arguments detailed here for two-dimensional situations extend to three dimensions as well.

One may notice that equations (3) are not discussed or verified a posteriori in the developments of [1]; they are treated as a priori requirements, then implicitly assumed to be satisfied by the proposed fundamental solution. Equations (3) are tested numerically in an example given in the end of the paper, but (i) the formulation is presented in [1] as an exact statement, which truth thus cannot established through numerical examples, and (ii) it has been pointed out above that such comparison can be made arbitrarily favorable by adjusting the value of the additive constant in $T(\boldsymbol{x})$.

The comments made herein address only the mathematical exactness of the formulation being examined; they are not meant to imply or suggest that the latter is without merit as an approximation.

References

[1] Kassab, A., Divo, E. A generalized boundary integral solution for heat conduction problems in non-homogeneous media. *Engng. Anal. with Bound. Elem.*, **18**, 273–286 (1996).